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A semi-infinite random walk associated with the game of roulette

M A El-Shehawey

Department of Mathematics, Damietta Faculty of Science, University of El-Mansoura, PO Box 6, New Damietta, Egypt

E-mail: EL_Shehawy@mum.mans.eun.eg

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Abstract

This paper is concerned with a discrete time random walk on the integers $0, 1, 2, \ldots$ which arises in the game of roulette. At each step either a unit displacement to the left with probability $1 - p$ or a fixed multiple displacement to the right with probability p can occur. There is a partially absorbing barrier at the origin, the probabilities of reflection and absorption at 0 are ρ and $1 - \rho$, respectively. Using generating functions and Lagrange's theorem for the expansion of a function as a power series, explicit expression for the probabilities of the player's capital at the nth step are deduced, as well as the probabilities of ultimate absorption at the origin.

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1. Introduction

Consider a one-dimensional discrete random walk over the possible positions $x = 0, 1, 2, \ldots,$ in which a particle can move a unit step to the left or m steps to the right. Let us introduce the following assumptions:

(i) The probability of a move to the left is $1 - p$, and consequently the probability of a move to the right is p, $p > 0$.

(ii) When the particle reaches the boundary point $x = 0$, it is absorbed with probability $1 - \rho$ and reflects (to the point $x = m$) with probability ρ, $0 \leqslant \rho \leqslant 1$.

(iii) The particle starts at the point N, $N \geqslant 0$.

This corresponds to the situation when, at the boundary 0, the particle is either lost from the system or turned back, and reduces to the classical problems of random walk, associated with the game of roulette, in the presence of absorbing or reflecting barriers at 0 for $\rho = 0$ and 1, respectively.

Anderson and Fontenot (1980) indicate that there is no closed form solution of the random walk model of the game of roulette in the presence of the assumptions that the casino is infinitely rich, the particle represents the player's capital, and at the origin there is a perfectly absorbing barrier. For a detailed description of the game and for actual values of m and p (see, for example, Epstein (1967) (p 132), and Downton and Holder (1972)). We are concerned in this paper with deriving an explicit expression for the probabilities $W_k^{(n)}$, $k = 0, 1, 2, \ldots$ of the particle being at the position k after n steps (i.e. the probabilities of the player's capital after the nth step). These are easily obtained from the generating functions and Lagrange's theorem for the expansion of a function as a power series. We remark that the present expressions for $W_k^{(n)}$, $k = 0, 1, 2, \ldots$ appear not to be readily available in the literature. It has previously been applied to particular cases (see, for example, Kac (1954) (p 295), Feller (1968) (p 352), Hill and Gulati (1981), Percus (1985) and El-Shehawey (2000)).

The outline of this paper is as follows. A system of equations for the generating function of the probabilities, $W_k^{(n)}$, $k = 0, 1, 2, \ldots$ are given in section 2. In section 3, using Lagrange's theorem, an explicit expression for the probability of ruin exactly at the nth step, $W_0^{(n)}$, is presented, as well as the probability of still playing after the nth step, for $n \geqslant N$. In section 4, we found an exact solution of the system and explicit expressions for the probabilities, $W_k^{(n)}$, $k = 0, 1, 2, \ldots$ the player's capital is k at the nth step are obtained. The probabilities of ultimate absorption, and the classical forms when $m = 1$ are given in section 5.

2. Generating function for the nth step probability of the player's capital

Let $W_k^{(n)}$, $k = 0, 1, 2, \ldots$ be the n-step probability that the particle is at location k after n steps (the player's capital is k after the nth step). The difference equations satisfied by $W_k^{(n)}$ for $n \geqslant 1$ are as follows:

$$W_k^{(n)} = \begin{cases} (1-p)\, W_{k+1}^{(n-1)} & \text{for } k = 0, 1, 2, \ldots, m-1 \\ (1-p)\, W_{k+1}^{(n-1)} + \rho\, W_{k-m}^{(n-1)} & \text{for } k = m \\ (1-p)\, W_{k+1}^{(n-1)} + p\, W_{k-m}^{(n-1)} & \text{for } k = m+1, m+2, \ldots \end{cases} \tag{2.1}$$

with the initial condition

$$W_k^{(0)} = \delta_{k,\,N} \qquad \text{for } N \geqslant 0 \tag{2.2}$$

where $\delta_{k,\,N}$ denotes the usual Kronecker delta.

Introducing the probability generating function $G_k(s)$, $k = 0, 1, 2, \ldots$

$$G_k(s) = \sum_{n=0}^{\infty} W_k^{(n)} s^n \qquad |s| < 1. \tag{2.3}$$

Multiplying (2.1) by s^n and summing over $n = 1, 2, \ldots$ and using (2.2) and (2.3), we obtain

$$G_k(s) = \delta_{k,N} + s \begin{cases} (1-p)\, G_{k+1}(s) & \text{for } k = 0, 1, \ldots, m-1 \\ \{(1-p)\, G_{k+1}(s) + \rho G_{k-m}(s)\} & \text{for } k = m \\ \{(1-p)\, G_{k+1}(s) + p\, G_{k-m}(s)\} & \text{for } k = m+1, m+2, \ldots. \end{cases} \tag{2.4}$$

From formula (2.4) and $Q(s, \lambda)$ defined by

$$Q(s,\ \lambda) = \sum_{k=0}^{\infty} \lambda^k\, G_k(s) \qquad \text{for } |\lambda| < 1 \tag{2.5}$$

it is a straightforward matter to deduce that

$$Q(s,\ \lambda) = \frac{\lambda^N - s\,(\alpha(\lambda) - \rho\,\lambda^m)\, G_0(s)}{1 - s\,\alpha(\lambda)} \tag{2.6}$$

where

$$\alpha(\lambda) = p\,\lambda^m + \frac{1-p}{\lambda}. \tag{2.7}$$

We observe that $Q(s, \lambda)$ is analytic when $|s| \leqslant 1$ and $|\lambda| < 1$. This implies that

$$\begin{aligned} G_0(\alpha^{-1}(\lambda)) &= \frac{\lambda^N}{1 - \rho\, s\, \lambda^m} \\ &= \sum_{r=0}^{\infty} (\rho\, s)^r\, \lambda^{N+m\,r} \qquad |\rho\, s\, \lambda^m| < 1 \end{aligned} \tag{2.8}$$

where s satisfies the relation $s\alpha(\lambda) = 1$. We then see from equation (2.7) that we need to determine a root λ of

$$\lambda = s(1-p) + s\,p\,\lambda^{m+1} \tag{2.9}$$

which is smaller than unity. There are two situations to consider, namely $|s| < 1$ and $|s| = 1$. Notice that we have to calculate functions $f(\lambda)$ of the form $f(\lambda = \lambda^x)$ where x is a positive integer, see equation (2.8). For $|s| < 1$, this can be easily calculated by a formal application of Lagrange's theorem (Whittaker and Watson (1963) (p 132)). If $\lambda = A + B\theta(\lambda)$ then

$$f(\lambda) = f(A) + \sum_{L=1}^{\infty} \frac{B^L}{L\,!}\, \frac{\mathrm{d}^{L-1}}{\mathrm{d}A^{L-1}} \{f'(A)\,[\theta(A)]^L\} \tag{2.10}$$

where f' is the first derivative of f. In this case if the inequality

$$|p\,s\,\lambda^{m+1}| < |\lambda - s\,(1-p)| \tag{2.11}$$

is satisfied for all points on the unit circle $|\lambda| = 1$ then $\alpha^{-1}(\lambda) = s$ has precisely one root in the interior of the unit circle, which is real for s real, $|s| < 1$. Using (2.10) λ is given by

$$\lambda = \frac{1}{\sqrt[m]{p\,s}} \int_0^{(1-p)\,\sqrt[m]{p\,s^{m+1}}} \sum_{L=0}^{\infty} \binom{(m+1)\,L}{L} x^{m\,L}\, \mathrm{d}x. \tag{2.12}$$

In order to show that (2.11) is satisfied for $|s| < 1$ and $|\lambda| = 1$ we take

$$\lambda = \mathrm{e}^{\mathrm{i}t} \qquad \text{and} \qquad s = w\mathrm{e}^{\mathrm{i}\varphi} \tag{2.13}$$

and (2.11) becomes

$$w^2(1-2p) + 1 > 2w(1-p)\cos(t-\varphi). \tag{2.14}$$

This is certainly the case if we consider w such that

$$w^2(1-2p) + 1 > 2w(1-p). \tag{2.15}$$

Inequality (2.15) can be simplified to

$$(1-w)[1 - w(1-2p)] > 0 \tag{2.16}$$

and since $1 - 2p < 1$ the restriction $|s| < 1$ (or $w < 1$) is a sufficient condition for (2.16) to hold and therefore (2.11). Thus for $|s| < 1$ the results obtained are valid for all p. This is not the case for $|s| = 1$. Consider, for example, the situation $s = 1$. In this case the equation $\alpha(\lambda) = 1$ admits the factor $(1 - \lambda)$ and (2.7) becomes

$$1 = p\,\lambda^m + \frac{1-p}{\lambda}$$

or

$$p \sum_{x=0}^{m} \lambda^x = 1 \tag{2.17}$$

and has a single root λ^* inside the unit circle if $(1+m)p > 1$ and no roots inside the unit circle if $(1+m)p \leqslant 1$. For all p there is a root at $\lambda = 1$ and a double root at $\lambda = 1$ if $(1+m)p = 1$.

3. Probabilities that the player's capital is zero at the nth step

In order to obtain an expression for λ^x, for any positive integer x, we can use formula (2.10) with $f(\lambda) = \lambda^x$, $A = s(1-p)$, $B = sp$ and $\theta(\lambda) = \lambda^{m+1}$, to obtain

$$\lambda^x = (s\,(1-p))^x \sum_{L=0}^{\infty} \frac{x}{x+(m+1)\,L} \binom{x+(m+1)\,L}{L} (p\,(1-p)^m\, s^{m+1})^L \qquad x \geqslant 1. \quad (3.1)$$

Equations (2.8) and (3.1) enable us to find the probability generating function $G_0(s)$ of the form

$$G_0(s) = (s\,(1-p))^N \sum_{L=0}^{\infty} \sum_{r=0}^{L} (\rho/p)^r \frac{N+mr}{N+(m+1)\,L-r} \times \binom{N+(m+1)\,L-r}{L-r} (p\,(1-p)^m\, s^{m+1})^L. \quad (3.2)$$

Observing (2.3) and (3.2) we can conclude that the probabilities that the player's capital is zero at the nth step are given by

$$W_0^{(n)} = \frac{N}{N+(m+1)\,L} \binom{N+(m+1)\,L}{L} p^L\,(1-p)^{N+mL} + \sum_{r=1}^{L} (\rho/p)^r \frac{N+mr}{N+(m+1)\,L-r} \binom{N+(m+1)\,L-r}{L-r} p^L\,(1-p)^{N+mL}$$
$$\text{for } n = N+(m+1)L,\ L \geqslant 0, = 0, \ \text{otherwise}. \quad (3.3)$$

Clearly the player's capital is zero at the nth step can occur only at steps $N+(m+1)L$ for $L \geqslant 0$.

Notice that $1-(1-\rho)\sum_{k=0}^{n} W_0^{(k)}$, denoted by U_0, is the probability of the player still playing after the nth step, for $n \geqslant N$, using (3.3) we obtain

$$U_0 = 1-(1-\rho)\,(1-p)^N \sum_{L=0}^{[\frac{n-N}{m+1}]} \left\{ \frac{N}{N+(m+1)\,L} \binom{N+(m+1)\,L}{L} + \sum_{r=1}^{L} \left(\frac{\rho}{p}\right)^r \frac{N+mr}{N+(m+1)\,L-r} \binom{N+(m+1)\,L-r}{L-r} \right\} (p\,(1-p)^m)^L \quad (3.4)$$

where $[y]$ denotes the greatest integer not greater than y. Clearly if $n < N$ then $U_0 = 1$. We remark that the player's capital is allowed to start from zero, i.e. $N = 0$, in this case the probabilities of the player's capital starting from zero and returning to zero at the nth step occur only at steps $(m+1)L$ for $L \geqslant 0$:

$$W_0^{((m+1)\,L)} = (p\,(1-p)^m)^L \sum_{r=1}^{L} \frac{mr}{(m+1)\,L-r} \binom{(m+1)\,L-r}{mL} \left(\frac{\rho}{p}\right)^r \qquad \text{for } L \geqslant 0. \quad (3.5)$$

4. Probabilities that the player's capital is k after the nth step

To calculate the probabilities that the player's capital is k after the nth step, $W_k^{(n)}, k = 1, 2, 3, \ldots$, it is convenient to rewrite (2.6) alternatively as

$$Q(s,\lambda) = G_0(s) + \frac{\lambda^N - (1 - s\,\rho\,\lambda^m)\,G_0(s)}{1 - s\,\alpha(\lambda)}. \quad (4.1)$$

Assuming $|s\alpha(\lambda)| < 1$, we obtain from (2.5) and (4.1), by evaluating the coefficients of s^n that

$$\sum_{k=0}^{\infty} W_k^{(n)}\lambda^k = \left(1 - \frac{\rho\lambda^m}{\alpha(\lambda)}\right) W_0^{(n)} + \lambda^N(\alpha(\lambda))^n - (1 - (\alpha(\lambda))^{-1}\rho\lambda^m) \sum_{j=0}^{n} W_0^{(j)}(\alpha(\lambda))^{n-j} \tag{4.2}$$

where the probabilities that the player's capital is zero at the nth step are given in (3.3).

Therefore, $W_k^{(n)}$ for $k = 1, 2, 3, \ldots$ are given by

$$W_k^{(n)} = \frac{1}{2\pi} \int_0^{2\pi} \Bigg\{ \mathrm{e}^{\mathrm{i}tN}(\alpha(\mathrm{e}^{\mathrm{i}t}))^n - (1 - \rho\,\mathrm{e}^{\mathrm{i}tm}(\alpha(\mathrm{e}^{\mathrm{i}t}))^{-1}) \\ \times \sum_{L=0}^{[\frac{n-N}{m+1}]} W_0^{(N+(m+1)L)}(\alpha(\mathrm{e}^{\mathrm{i}t}))^{n-N-(m+1)L} \Bigg\} \mathrm{e}^{-\mathrm{i}kt}\,\mathrm{d}t. \tag{4.3}$$

It is a simple matter to show from (2.7) that for any two integers β and γ

$$\frac{1}{2\pi}\int_0^{2\pi} (\alpha(\mathrm{e}^{\mathrm{i}t}))^{\beta}\,\mathrm{e}^{-\mathrm{i}\gamma t}\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} \{p\,\mathrm{e}^{\mathrm{i}mt} + (1-p)\,\mathrm{e}^{-\mathrm{i}t}\}^{\beta}\,\mathrm{e}^{-\mathrm{i}\gamma t}\,\mathrm{d}t \\ = \binom{\beta}{\frac{m\beta-\gamma}{m+1}} p^{\frac{\beta+\gamma}{m+1}}\,(1-p)^{\frac{m\beta-\gamma}{m+1}} \tag{4.4}$$

provided that $(m\,\beta - \gamma)/(m+1)$ is an integer.

From (4.3) and (4.4) the probabilities $W_k^{(n)}$ for fixed k are found to be

$$W_k^{(n)} = \binom{N+(m+1)\,j-k}{j} p^j\,(1-p)^{N+m\,j-k} \\ - \sum_{L=0}^{[j-\frac{k+1}{m+1}]} \left(1 - \frac{\rho}{p}\frac{(j-L)}{(m+1)(j-L)-k}\right) W_0^{(N+(m+1)L)} \\ \times \binom{(m+1)\,(j-L)\,-k}{j-L} p^{j-L}\,(1-p)^{m\,(j-L)-k} \\ \text{for } n = N+(m+1)j-k,\ j \geqslant 0 \text{ and } W_k^{(n)} = 0, \text{ otherwise} \tag{4.5}$$

where $[y]$ is to be taken as the greatest integer not greater than y if y is non-negative. If $j - (k+1)/(m+1)$ is strictly negative then the term involving the summation in (4.5) does not arise.

Formulae (3.3) and (4.5) enable us to find explicit expressions for the only non-zero probabilities $W_k^{(n)}$, $k = 0, 1, 2, 3, \ldots$ in the form

$$W_k^{(n)} = p^j\,(1-p)^{N+m\,j-k} \Bigg\{ \binom{N+(m+1)\,j-k}{j} \\ - \sum_{L=0}^{[j-\frac{k+1}{m+1}]} \sum_{r=0}^{L} \left(\frac{\rho}{p}\right)^r \left(1 - \frac{\rho}{p}\frac{(j-L)}{(m+1)(j-L)-k}\right) \\ \times \frac{N+mr}{N+(m+1)L-r} \binom{N+(m+1)L-r}{L-r} \binom{(m+1)\,(j-L)\,-k}{j-L} \Bigg\} \\ \text{for } n = N+(m+1)j-k,\ j \geqslant 0 \text{ and } W_k^{(n)} = 0 \text{ otherwise.} \tag{4.6}$$

Notice that the first term of (4.6) (or (4.5)) is the unconditional probability that the player's capital is k after $N + (m+1)j - k$ steps with no partially absorbing barrier at the origin. The subtracted summation term arises from the possibility of absorption, with probability $(1-\rho)$, prior to the $(N + (m+1)j - k)$th step. Each term involves the probability the player's capital is k with a passage to $k = 0$ at step $N + (m+1)L - r$ times the unconditional probability that the player's capital starting from 0 and arriving at k in $(m+1)(j-L) - k$ steps.

If a perfectly absorbing barrier at the origin is considered, i.e. $\rho = 0$, then the probability that the player's capital is k after the nth step for $j \geqslant 0$ is given by

$$W_k^{(N+(m+1)\,j-k)} = p^j (1-p)^{N+m\,j-k} \times \begin{cases} \dfrac{N}{N+(m+1)\,j} \dbinom{N+(m+1)\,j}{j} & \text{for } k = 0 \\ \dbinom{N+(m+1)j-k}{j} - \displaystyle\sum_{L=0}^{[j-\frac{k}{m+1}]} \dfrac{N}{N+(m+1)L} & \\ \times \dbinom{N+(m+1)\,L}{L} \dbinom{(m+1)\,(j-L)\,-k}{j-L} & \text{for } k = 1,\ 2,\ 3, \ldots. \end{cases} \tag{4.7}$$

Formula (4.7) can alternatively be expressed as

$$W_k^{(n)} = \begin{cases} \dfrac{N}{n} \dbinom{n}{\frac{n-N}{m+1}} p^{\frac{n-N}{m+1}} (1-p)^{\frac{m\,n+N}{m+1}} & \text{for } k = 0 \\ \left[\dbinom{n}{\frac{n-N+k}{m+1}} - \dbinom{n}{\frac{n+N+k}{m+1}} \right] p^{\frac{n-N+k}{m+1}} (1-p)^{n-(\frac{n-N+k}{m+1})} & \text{for } k = 1, 2, \ldots. \end{cases} \tag{4.8}$$

We see that with the appropriate change of notation, expression (4.7) agrees with that of Hill and Gulati (1981), formulae (14) and (21); expression (4.8) agrees with that of Percus (1985) in the classical case $m = 1$ (see, also, Feller (1968) (p 352)).

5. The probability of absorption at the boundary

In order to deduce an expression for the probabilities of ultimate absorption at the boundary 0, we first find the expected value of the number of times the state 0 is occupied, $E[n_N(m)]$, defined by

$$E[n_N(m)] = \sum_{n=0}^{\infty} W_0^{(n)} \tag{5.1}$$

(see, for example, Kemeny and Snell (1976) (pp 46, 47) and Iosifescu (1980) (pp 99, 100)), from formulae (2.8) and (2.12) after setting $s = 1$, we obtain for $N \geqslant 0$,

$$E[n_N(m)] = \begin{cases} \dfrac{\lambda^N}{1-\rho\,\lambda^m} & \text{for } p > \frac{1}{m+1} \\ \dfrac{1}{1-\rho} & \text{for } p \leqslant \frac{1}{m+1} \end{cases} \tag{5.2}$$

where

$$\lambda = \frac{1}{\sqrt[m]{p}} \int_0^{(1-p)\ \sqrt[m]{p}} \left[\sum_{L=0}^{\infty} \binom{(m+1)\,L}{L} x^{m\,L} \right] \mathrm{d}x. \tag{5.3}$$

Thus the probabilities of ultimate absorption at 0, $(1-\rho)\,E[n_N(m)]$ are given by

$$\begin{cases} (1-\rho)\,\lambda^N\left\{1+\sum_{r=1}^{\infty}(\rho\,\lambda^m)^r\right\} & \text{for } p>\frac{1}{m+1} \\ 1 & \text{for } p\leqslant\frac{1}{m+1} \end{cases} \tag{5.4}$$

whereas the survival probabilities are therefore

$$\begin{cases} 1-(1-\rho)\,\lambda^N\sum_{r=0}^{\infty}(\rho\,\lambda^m)^r & \text{for } p>\frac{1}{m+1} \\ 0 & \text{for } p\leqslant\frac{1}{m+1}. \end{cases} \tag{5.5}$$

Many interesting special cases can be derived from our results through appropriate choices of m, ρ, N and p; an example is given below:

Consider the case $m=1$, using the identity

$$\sum_{L=0}^{\infty}\binom{2L}{L}x^L=(1-4x)^{-\frac{1}{2}} \tag{5.6}$$

formula (2.12) becomes

$$\lambda^*=\frac{1}{2ps}\left(1-\sqrt{1-4p\,(1-p)\,s^2}\right) \tag{5.7}$$

and the probability generating function expression for the player's capital is k after the nth step is given by

$$\sum_{n=0}^{\infty}W_0^{(n)}\,s^n=\frac{\lambda^{*N}}{1-s\,\rho\,\lambda^*}. \tag{5.8}$$

Therefore

$$E[n_N(1)]=\begin{cases} \dfrac{\left(\frac{1-p}{p}\right)^N}{1-\rho\left(\frac{1-p}{p}\right)} & \text{for } p>\frac{1}{2} \\ \dfrac{1}{1-\rho} & \text{for } p\leqslant\frac{1}{2} \end{cases} \tag{5.9}$$

and the probability of absorption at the boundary 0 takes the form

$$(1-\rho)\,E[n_N(1)]=\begin{cases} \dfrac{(1-\rho)}{1-\rho\left(\frac{1-p}{p}\right)}\left(\dfrac{1-p}{p}\right)^N & \text{for } p>\frac{1}{2} \\ 1 & \text{for } p\leqslant\frac{1}{2} \end{cases} \tag{5.10}$$

whereas

the survival probability is

$$\begin{cases} 1-\dfrac{(1-\rho)}{1-\rho\left(\frac{1-p}{p}\right)}\left(\dfrac{1-p}{p}\right)^N & \text{for } p>\frac{1}{2} \\ 0 & \text{for } p\leqslant\frac{1}{2}. \end{cases} \tag{5.11}$$

The mean number of steps taken until being absorbed, given that the player is absorbed at the boundary 0, E_N^*, is given for $\rho\neq 1$ by

$$\begin{aligned} E_N^* &= \frac{(1-\rho)\sum_{n=0}^{\infty}n\,W_0^{(n)}}{(1-\rho)\,E[n_N(1)]} = \frac{1}{E[n_N(1)]}\,\frac{\mathrm{d}}{\mathrm{d}s}\,G_0(s)\bigg|_{s=1} \\ &= \begin{cases} \dfrac{1}{|1-2p|}\left\{N+\dfrac{2\,\rho\,(1-p)}{1-\rho\min\left(1,\frac{1-p}{p}\right)}\right\} & \text{for } p\neq\frac{1}{2} \\ \infty & \text{for } p=\frac{1}{2}. \end{cases} \end{aligned} \tag{5.12}$$

We have the immediate consequence from formula (3.3) that the probability of the player's capital is zero at the nth step:

$$W_0^{(n)} = \begin{cases} \dfrac{N}{n}\dbinom{n}{\frac{n-N}{2}} p^{\frac{n-N}{2}}(1-p)^{\frac{n+N}{2}} & \text{for } \rho = 0 \\ p^{\frac{n-N}{2}}(1-p)^{\frac{n+N}{2}} \displaystyle\sum_{j=0}^{[\frac{n-N}{2}]} \frac{n+N-2j}{n+N+2j}\dbinom{\frac{n+N}{2}+j}{j} & \text{for } \rho = p \\ p^{\frac{n-N}{2}}(1-p)^{\frac{n+N}{2}} \displaystyle\sum_{j=0}^{[\frac{n-N}{2}]} \frac{n+N-2j}{n+N+2j}\dbinom{\frac{n+N}{2}+j}{j} 2^{\frac{n-N}{2}-j} & \text{for } \rho = 2p \\ (1-p)^{\frac{n+N}{2}} \displaystyle\sum_{j=0}^{[\frac{n-N}{2}]} \frac{n+N-2j}{n+N+2j}\dbinom{\frac{n+N}{2}+j}{j} p^j & \text{for } \rho = 1. \end{cases} \tag{5.13}$$

Similar expressions can be obtained for the probability of the player's capital is $k, k = 1, 2, \ldots$ at the nth step using formula (4.6). These expressions show that $W_k^{(n)}, k = 0, 1, \ldots$ takes on qualitatively different forms depending on the relationship between p and ρ.

We see that with the appropriate change of notation, formula (5.4) with $\rho = 0$ agrees with that of Hill and Gulati (1981). Formulae (5.7)–(5.12) agree with that of Percus (1985) (see, also, El-Shehawey (2000)).

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